

Solution to Problem Set 4

1. $\psi(x) = Cx \sin\left(\frac{n\pi x}{L}\right);$

Normalization :

$$\int_0^L \psi^* \psi dx = 1$$

$$\Rightarrow C^2 \int_0^L x^2 \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$

$$\Rightarrow C^2 \left[\frac{x^3}{6} - \frac{L^2 x \cos\left(\frac{2n\pi x}{L}\right)}{4n^2 \pi^2} - \frac{L(-L^2 + 2n^2 \pi^2 x^2) \sin\left(\frac{2n\pi x}{L}\right)}{8n^3 \pi^3} \right]_0^L = 1$$

$$\Rightarrow C^2 \left[\frac{L^3}{6} - \frac{L^3}{4n^2 \pi^2} \right] = 1 \Rightarrow C = \sqrt{\frac{1}{\frac{L^3}{6} - \frac{L^3}{4n^2 \pi^2}}}$$

E =

$$\int_0^L \psi^* H \psi dx = \int_0^L Cx \sin\left(\frac{n\pi x}{L}\right) * -\frac{h^2}{8m\pi^2} \frac{d^2 \psi}{dx^2} dx$$

$$= \int_0^L Cx \sin\left(\frac{n\pi x}{L}\right) * -\frac{h^2}{8m\pi^2} \left[\frac{2n\pi C}{L} \cos\left(\frac{n\pi x}{L}\right) - \frac{Cn^2 \pi^2}{L^2} x \sin\left(\frac{n\pi x}{L}\right) \right] dx$$

$$= \frac{C^2 h^2 L}{192m\pi^3} * (4n^3 \pi^3 + 6n\pi) = \frac{nh^2 (4n^3 \pi^3 + 6n\pi)}{16m\pi L^2 (2n^2 \pi^2 - 3)}$$

$$= \frac{h^2 n^2 (2n^2 \pi^2 + 3)}{8mL^2 (2n^2 \pi^2 - 3)} = \frac{h^2 (2n^4 \pi^2 + 3n^2)}{8mL^2 (2n^2 \pi^2 - 3)} = \frac{(2n^4 \pi^2 + 3n^2)}{(2n^2 \pi^2 - 3)} * E_1$$

For n = 6, E = 36.3053 * E₁

2. For $s \neq n$,

$$\begin{aligned}
 a_s &= \int_0^L Cx \sin\left(\frac{s\pi x}{L}\right) * \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx \\
 &= \left[-\frac{L^{\frac{3}{2}}}{\pi^2 (n-s)^2 (n+s)^2} \left(\sqrt{2}C(2ns - 2ns \cos(n\pi) \cos(s\pi)) \right) \right] \\
 &= \left[\frac{4\sqrt{6}n^2 s}{\pi\sqrt{2n^2\pi^2 - 3}(n-s)^2 (n+s)^2} (\cos(n\pi) \cos(s\pi) - 1) \right] \\
 &= \frac{4\sqrt{6}n^2 s}{\pi\sqrt{2n^2\pi^2 - 3}(n-s)^2 (n+s)^2} \left((-1)^{n+s} - 1 \right) \\
 P_s = a_s^2 &= \frac{96n^4 s^2 (-1 + (-1)^{n+s})^2}{\pi^2 (2n^2\pi^2 - 3)(n-s)^4 (n+s)^4} \quad (1)
 \end{aligned}$$

For $s = n$,

$$\begin{aligned}
 a_s &= \int_0^L C \sqrt{\frac{2}{L}} x \sin^2\left(\frac{s\pi x}{L}\right) dx = \frac{CL^{\frac{3}{2}}}{2\sqrt{2}} = \frac{\sqrt{3}\pi s}{\sqrt{4\pi^2 s^2 - 6}} \\
 \Rightarrow P_s &= \frac{3\pi^2 s^2}{4\pi^2 s^2 - 6} \quad (2)
 \end{aligned}$$

A. In (1), for odd n , P_s (odd s) = 0 and for even n , P_s (even s) = 0 i.e. every second term is zero. Among the surviving terms, the states further away from n would make $(n - s)$ and $(n + s)$ larger & hence the probability would get smaller and smaller. The probability is maximum for $s = n, n - 1$ & $n + 1$. Therefore, for any n , the three states most likely to be populated are $n - 1, n$ and $n + 1$ & account for nearly 100% of the population since all the other terms are either zero or negligible compared to these 3 terms!

Total Probability of these 3 states

$$= \frac{384n^4}{\pi^2 (2n^2\pi^2 - 3)} \left(\frac{(n-1)^2}{(2n-1)^4} + \frac{(n+1)^2}{(2n+1)^4} \right) + \frac{3n^2\pi^2}{2(2n^2\pi^2 - 3)}$$

B. $n = 6$;

Assuming only $n - 1, n$ & $n + 1$ states contribute (to do this problem analytically!).

$$\begin{aligned}
 E &= \sum_s P_s E_s = \sum_s a_s^2 E_s = E_1 * \sum_s P_s s^2 \\
 &= E_1 * \left[\frac{384n^4}{\pi^2 (2n^2\pi^2 - 3)} \left(\frac{(n-1)^4}{(2n-1)^4} + \frac{(n+1)^4}{(2n+1)^4} \right) + \frac{3n^4\pi^2}{2(2n^2\pi^2 - 3)} \right] \\
 &= E_1 * (3.04194 + 5.99046 + 27.1145) = 36.1469 * E_1.
 \end{aligned}$$

C. Doing this summation over very large values of s , say 100, P_s converges to 1 & E converges to $36.3053 * E_1$ as in problem 1.

3. A. 2D box :

Consider the area of a quadrant with radius n in a space axes n_x and n_y .

$$N(E) = \frac{1}{4} \cdot \pi n^2$$

$$E = E_1 n^2$$

$$\Rightarrow n(E) = \frac{1}{4} \pi \left(\frac{E}{E_1} \right)$$

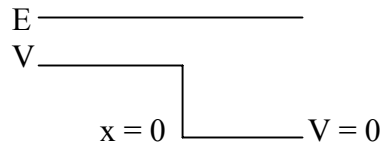
$$\Rightarrow \rho(E) = \frac{dN(E)}{dE} = \frac{1}{4} * \frac{\pi}{E_1} = \frac{0.25\pi}{E_1}$$

B. 1D well :

$$N(E) = n = \sqrt{\frac{E}{E_1}}$$

$$\rho(E) = \frac{dN(E)}{dE} = \frac{1}{2} * \sqrt{\frac{1}{EE_1}} = \frac{0.5}{\sqrt{E_1}} (E)^{-\frac{1}{2}}$$

4.



Consider the wave incident from the left !

$$x < 0, \quad \psi(x) = Ae^{ikx} + Be^{-ikx} \text{ where } k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$x > 0, \quad \psi(x) = Ce^{ik'x} + De^{-ik'x} \text{ where } D = 0 \text{ \& } k' = \sqrt{\frac{2m(E-V)}{\hbar^2}}$$

For $x = 0$, $A + B = C$

Continuity at $x = 0$ implies $k(A - B) = k' C$

$$\Rightarrow \frac{A - B}{C} = \frac{A}{C} - \frac{B}{C} = 1 - \frac{2B}{C} = \frac{k'}{k}$$

$$\Rightarrow \frac{B}{C} = \frac{k - k'}{2k} \Rightarrow \frac{A}{C} = 1 - \frac{k - k'}{2k} = \frac{k + k'}{2k}$$

$$\Rightarrow \frac{C}{A} = \frac{2k}{k + k'}$$

$$\frac{B}{A} = \frac{C}{A} - 1 = \frac{2k}{k + k'} - 1 = \frac{k - k'}{k + k'}$$

$$\text{Probability that it is reflected} = \left| \frac{B}{A} \right|^2 = \left(\frac{k - k'}{k + k'} \right)^2$$

$$\text{Probability that it passes over the step} = \frac{k'}{k} \left| \frac{C}{A} \right|^2 = \frac{4kk'}{(k + k')^2}$$

$$m = 1.66 * 10^{-27} \text{ kg ; } E = 5 \text{ eV ; } V = 4 \text{ eV ;}$$

$$k = \sqrt{\frac{8m\pi^2 E}{h^2}} = \sqrt{\frac{8 * 1.66 * 10^{-27} * \pi^2 * 5 * 1.6 * 10^{-19}}{(6.626 * 10^{-34})^2}} = 4.887 * 10^{11}$$

$$k' = \sqrt{\frac{8m\pi^2 (E - V)}{h^2}} = \sqrt{\frac{8 * 1.66 * 10^{-27} * \pi^2 * 1 * 1.6 * 10^{-19}}{(6.626 * 10^{-34})^2}} = 2.186 * 10^{11}$$

Therefore, R = 0.1459 and T = 0.8542

Problems from Levine :

2.18.a. $\psi_I(x) = Ce^{kx}$; $\psi_{II}(x) = A \cos(k'x) + B \sin(k'x)$ where

$$k = \sqrt{\frac{2m(V_o - E)}{\hbar^2}} \text{ and } k' = \sqrt{\frac{2mE}{\hbar^2}}$$

We already know that C = A !

$$\psi'_I(0) = \psi'_{II}(0)$$

$$\Rightarrow Cke^{kx} = k'(B \cos k'x - A \sin k'x)$$

$$\Rightarrow Ck = Bk'$$

$$\Rightarrow \frac{k}{k'} = \frac{B}{C} = \frac{B}{A} = \sqrt{\frac{V_o - E}{E}}$$

$$\text{i.e. } B = A \sqrt{\frac{V_o - E}{E}}$$

b.

$$\psi_{II}(l) = \psi_{III}(l) \tag{1}$$

$$\Rightarrow A \cos k'l + B \sin k'l = Ge^{-kl}$$

$$\psi'_{II}(l) = \psi'_{III}(l) \tag{2}$$

$$\Rightarrow k'(A \sin k'l - B \cos k'l) = Gke^{-kl}$$

Dividing (2) by (1), we get :

$$\frac{k}{k'} = \frac{A \sin k'l - B \cos k'l}{A \cos k'l + B \sin k'l}$$

Dividing the right hand side by A cos k'l, we get

$$\frac{k}{k'} = \frac{B}{A} = \frac{\tan k'l - \frac{B}{A}}{1 + \frac{B}{A} \tan k'l}$$

$$\Rightarrow \frac{2B}{A} = \tan k'l \left(1 - \left(\frac{B}{A}\right)^2\right)$$

$$\Rightarrow 2\sqrt{\frac{V_o - E}{E}} = \frac{\sin k'l \left(\frac{2E - V_o}{E}\right)}{\cos k'l}$$

$$\Rightarrow (2E - V_o) \sin\left(\sqrt{\frac{2mE}{\hbar^2}}l\right) = 2\sqrt{V_o E - E^2} \cos\left(\sqrt{\frac{2mE}{\hbar^2}}l\right)$$

2.21. $V_o = 15\text{eV}$; $L = 2 \text{ \AA}$; $m = 9.1 \cdot 10^{-31} \text{ kg}$;

$$b = \sqrt{2mV_o} \frac{2\pi L}{h} = \sqrt{2 \cdot 9.1 \cdot 10^{-31} \cdot 15 \cdot 1.6 \cdot 10^{-19}} \cdot \frac{2\pi \cdot 2 \cdot 10^{-10}}{6.626 \cdot 10^{-34}} = 3.96$$

Therefore, $\frac{b}{\pi} = 1.26$

$$N - 1 < \frac{b}{\pi} \leq N$$

$$\Rightarrow N - 1 < 1.26 \Rightarrow N < 2.26$$

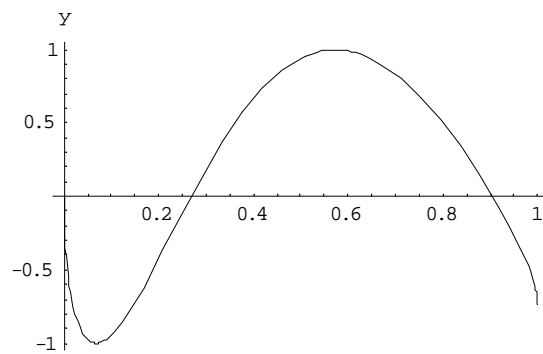
Also $N \geq 1.26$

Therefore, $N = 2$

2.22. $b = 3.96$;

$$\text{Let } y = (2\varepsilon - 1)\sin(b\sqrt{\varepsilon}) - 2\sqrt{\varepsilon - \varepsilon^2} \cos(b\sqrt{\varepsilon})$$

Plotting y as a function of ε where $0 < \varepsilon < 1$, we get



On zooming in on the points of intersection, we get $\varepsilon = 0.905185$ and $\varepsilon = 0.268536$.

From $\varepsilon = E/V_o$, we get the allowed bound state energies to be $E = 13.58\text{eV}$ and $E = 4.03 \text{ eV}$.