<u>**Solution to Problem Set 2**</u>

1.

$$|u\rangle = 2|1\rangle + 5|2\rangle + \alpha|3\rangle$$
$$|v\rangle = |1\rangle + 2|2\rangle - 3|3\rangle$$

A.

$$\langle u|v\rangle = 0$$
$$\Rightarrow (2\langle 1| + 5\langle 2| + \alpha\langle 3|) * (|1\rangle + 2|2\rangle - 3|3\rangle) = 0$$
$$i.e. 2 + 10 - 3\alpha = 0$$

i.e. **α = 4**

B.

$$C^2\langle u|u\rangle = 1$$
$$\Rightarrow C^2(4 + 25 + 16) = 1$$
$$\Rightarrow C^2 = \frac{1}{45}$$

i.e. Normalized vector $|u\rangle = \frac{1}{3\sqrt{5}}|u\rangle$

and Normalized vector $|v\rangle = \frac{1}{\sqrt{14}}|u\rangle$

2. $f(x) = \sum_n a_n \phi_n(x), a \le x \le b$

A.

$$\therefore \int_a^b r(x)\cdot[f(x)]^2 dx = \sum_n a_n \sum_m a_m \int_a^b r(x)\cdot\phi_n(x)\cdot\phi_m(x)dx = \sum_n a_n \sum_m a_m \cdot \delta_{nm} = \sum_n a_n^2$$

B. f(x) = x ; r(x) = 1 ; 0 ≤ x ≤ L

$$\phi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \int_0^L r(x)\cdot f(x)\cdot\phi_n(x)dx = \sqrt{\frac{2}{L}}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \sqrt{\frac{2}{L}}\cdot\frac{L}{n^2\pi^2}\left[L\sin\left(\frac{n\pi x}{L}\right) - n\pi x\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$\sqrt{\frac{2}{L}}\frac{L^2}{n^2\pi^2}(\sin n\pi - n\pi\cos n\pi) = \sqrt{\frac{2}{L}}\frac{L^2}{n^2\pi^2}\left(-n\pi(-1^n)\right) = -\frac{\sqrt{2}L^{\frac{3}{2}}}{n\pi}(-1)^n$$

From Part A, we know that $\int_0^L x^2 dx = \frac{L^3}{3} = \sum_n a_n^2$

Therefore,

$$\frac{L^3}{3} = \sum_n \frac{2L^3}{n^2\pi^2}$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_n \frac{1}{n^2}$$

i.e. $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

3. f(x) = 1 ; 0 ≤ x ≤ L ;

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \int_0^L r(x)\cdot f(x)\cdot \phi_n(x)dx = \sqrt{\frac{2}{L}}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx = -\sqrt{\frac{2}{L}}\cdot\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)_0^L$$

$$= -\frac{\sqrt{2L}}{n\pi}\left[\cos(n\pi)-1\right] = \frac{\sqrt{2L}}{n\pi}\left[1-(-1)^n\right]$$

$$\text{f(x)} = 1 = \sum_n a_n\phi_n(x) = \frac{2}{\pi}\sum_n \frac{1-(-1)^n}{n}\sin\left(\frac{n\pi x}{L}\right)$$

$$\therefore 1 = \frac{4}{\pi}\left[\frac{1}{1}\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{L}\right) + \dots\right]$$

We know that L = $\sum_n a_n^2$

$$\therefore L = \sum_n \frac{2L}{n^2\pi^2}\left[2-2(-1)^n\right] = \sum_n \frac{4L}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$\Rightarrow \frac{\pi^2}{4} = 2\cdot\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\right]$$

i.e. $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

4. f(x) = x^2 ; 0 ≤ x ≤ L ;

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \int_0^L r(x)\cdot f(x)\cdot \phi_n(x)dx = \sqrt{\frac{2}{L}}\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \sqrt{\frac{2}{L}}\cdot\frac{L}{n^3\pi^3}\left[2Ln\pi x\sin\left(\frac{n\pi x}{L}\right) + (2L^2 - n^2\pi^2x^2)\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \sqrt{\frac{2}{L}}\cdot\frac{L}{n^3\pi^3}\left[\left(2L^2 - n^2\pi^2L^2)(-1)^n\right) - 2L^2\right]$$

$$\sum_n a_n\phi_n(x) = \frac{2}{\pi^3}\sum_n \frac{\left[\left(2L^2 - n^2\pi^2L^2)(-1)^n\right) - 2L^2\right]}{n^3}\sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2}{\pi^3}\left[(-4L^2 + \pi^2L^2)\sin\left(\frac{\pi x}{L}\right) + (\frac{-\pi^2L^2}{2})\sin\left(\frac{2\pi x}{L}\right) + (\frac{-4L^2 + 9\pi^2L^2}{27})\sin\left(\frac{3\pi x}{L}\right) + \ldots\right]$$

Therefore,

$$\mathbf{x^2} = \frac{2L^2}{\pi^3}\left[(-4+\pi^2)\sin\left(\frac{\pi x}{L}\right) + (\frac{-\pi^2}{2})\sin\left(\frac{2\pi x}{L}\right) + (\frac{-4+9\pi^2}{27})\sin\left(\frac{3\pi x}{L}\right) + \ldots\right]$$

5. $\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2ny = 0\,,\ -\infty \le x \le \infty$

A. $a_0(x) = 1$; $a_1(x) = -2x$; $a_2(x) = 0$; $a_3(x) = \dfrac{2n}{\lambda}$;

$$p(x) = e^{\int \frac{a_1}{a_0}dx} = e^{\int -2x dx} = e^{-x^2}$$

$$q(x) = \frac{a_2}{a_0}p(x) = 0$$

$$r(x) = \frac{a_3}{a_0}p(x) = \frac{2ne^{-x^2}}{\lambda}$$

Therefore, the equation in Sturm-Liouville form is

$$\frac{d}{dx}\left[e^{-x^2}\frac{dy}{dx}\right] + 2ne^{-x^2}y = 0$$

B. $$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$
$$H_0(x) = 1;$$
$$H_1(x) = (-1)e^{x^2} e^{-x^2}(-2x) = 2x;$$
$$H_2(x) = e^{x^2} \frac{d}{dx}\left[-2xe^{-x^2}\right] = -2e^{x^2}\left[e^{-x^2} - 2x^2 e^{-x^2}\right] = 4x^2 - 2;$$
$$H_3(x) = -e^{x^2} \frac{d^2}{dx^2}\left[-2xe^{-x^2}\right] = 2e^{x^2} \frac{d}{dx}\left[e^{-x^2} - 2x^2 e^{-x^2}\right]$$
$$= 2e^{x^2}\left[4x^3 e^{-x^2} - 4xe^{-x^2} - 2xe^{-x^2}\right] = 8x^3 - 12x;$$

C.
$$xH_1(x) = H_0(x) + \frac{1}{2}H_2(x)$$
$$\Rightarrow x \cdot 2x = 1 + \frac{1}{2}(4x^2 - 2)$$
$$\Rightarrow 2x^2 = 2x^2$$
$$xH_2(x) = 2H_1(x) + \frac{1}{2}H_3(x)$$
$$\Rightarrow x(4x^2 - 2) = 2.2x + \frac{1}{2}(8x^3 - 12x)$$
$$\Rightarrow 4x^3 - 2x = 4x^3 - 2x$$

D. Assume 2n = λ, r(x) = e^{-x^2} ;
$$H_2(x) = 4x^2 - 2; H_3(x) = 8x^3 - 12x;$$
$$\therefore \int_{-\infty}^{\infty} H_2(x) \cdot H_3(x) r(x) dx = \int_{-\infty}^{\infty} e^{-x^2}(4x^2 - 2) \cdot (8x^3 - 12x) dx = \int_{-\infty}^{\infty} evenfn \cdot evenfn \cdot oddfn \cdot dx$$
$$= \int_{-\infty}^{\infty} oddfn \cdot dx = 0$$

Thus, $H_2(x)$ and $H_3(x)$ are orthogonal !

6. $$\Im|n\rangle = n^2|n\rangle; |f\rangle = \frac{1}{\sqrt{6}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle + \frac{1}{\sqrt{2}}|3\rangle;$$

A. The probability that the system will be found in state $|1\rangle$ = $|a_1|^2$

i.e. Probability = 1/6

B.

$$\langle L\rangle = \langle f|\Im|f\rangle = \frac{1}{\sqrt{6}}\langle 1| + \frac{1}{\sqrt{3}}\langle 2| + \frac{1}{\sqrt{2}}\langle 3|\left(\frac{1}{\sqrt{6}} * 1 + \frac{1}{\sqrt{3}} * 4 + \frac{1}{\sqrt{2}} * 9\right)$$

$$= \left[\frac{1}{6} + \frac{4}{3} + \frac{9}{2}\right] = 6$$

Therefore, the expectation value of L = 6