Solutions to Problem Set 1

Problems from Levine :

1.1 a. λ = 1064 nm ; n = 1 ;

$$E = nh\nu = nh\frac{c}{\lambda} = 1*6.626*10^{-34}*\frac{3*10^{8}}{1064*10^{-9}}$$

= 1.87 * 10^{-19} J = 1.17 eV

b. Ave. Power = 5*10^{6} W ; τ = 2*10^{-8} s ; λ = 1064 nm ;

Ave. Power = Energy per Pulse

$$\Rightarrow E = Ave.Power * \tau$$

$$= 5*10^{6}*2*10^{-8} = 0.1J$$

$$\text{No. of Photons } n = \frac{E\lambda}{hc} = \frac{0.1*1064*10^{-9}}{6.626*10^{-34}*3*10^{8}}$$

= 5.35*10^{17}

1.4. a. w = 0.1 cm ; V = 1000 V ;

$$KE = eV = \frac{p^2}{2m}$$

$$\Rightarrow p = \sqrt{2meV} = \sqrt{2*9.1*10^{-31}*1.6*10^{-19}*1000} = 1.71*10^{-23}$$

$$\Rightarrow \lambda = \frac{h}{p} = \frac{6.626*10^{-34}}{1.71*10^{-23}} = 3.88*10^{-11}m$$

$$\Rightarrow \sin\alpha = \frac{\lambda}{w} \Rightarrow \alpha = \sin^{-1}(\frac{\lambda}{w}) = \sin^{-1}(\frac{6.626*10^{-34}}{0.1*10^{-2}})$$

Therefore,

α **= 2.23*10^{-6} degrees**

b. $\alpha = 1$;

$$w = \frac{\lambda}{\sin\alpha} = \frac{3.88*10^{-11}}{\sin 1}$$

Slit width w = 2.22 nm

1.10. t = 0 ; c = 2 A^o ;

$$\text{Probability} = \frac{\int_2^{2.001} \left(\frac{32}{\pi c^6}\right)^{1/4} x e^{-\frac{x^2}{c^2}} \bullet \left(\frac{32}{\pi c^6}\right)^{1/4} x e^{-\frac{x^2}{c^2}} dx}{\frac{1}{\sqrt{2\pi}} \int_2^{2.001} x^2 e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \left[\int_0^{2.001} x^2 e^{-\frac{x^2}{2}} dx - \int_0^2 x^2 e^{-\frac{x^2}{2}} dx \right]}$$

$$\text{Let } u^2 = \frac{x^2}{2} \Rightarrow x = \sqrt{2}u$$
$$dx = \sqrt{2}du$$

$$\therefore \int x^2 e^{-\frac{x^2}{2}} dx = 2\sqrt{2} \int u^2 e^{-u^2} du = \sqrt{2} \int u \cdot d(-e^{-u^2})$$
$$= -\sqrt{2}ue^{-u^2} + \sqrt{2} \int e^{-u^2} du$$
$$= -\sqrt{2}ue^{-u^2} + \sqrt{\frac{\pi}{2}} erf(u) = -xe^{-\frac{x^2}{2}} + \sqrt{\frac{\pi}{2}} erf(\frac{x}{\sqrt{2}})$$

Probability =

$$\frac{1}{\sqrt{2\pi}} \left[\int_0^{2.001} x^2 e^{-\frac{x^2}{2}} dx - \int_0^2 x^2 e^{-\frac{x^2}{2}} dx \right]$$

$$\frac{1}{\sqrt{2\pi}} \left[(-xe^{-\frac{x^2}{2}} \Big|_0^{2.001} + \sqrt{\frac{\pi}{2}} erf(\frac{2.001}{\sqrt{2}})) - (-xe^{-\frac{x^2}{2}} \Big|_0^2 + \sqrt{\frac{\pi}{2}} erf(\frac{2}{\sqrt{2}})) \right]$$

$$\frac{1}{\sqrt{2\pi}} \left[-2.001 * e^{-\frac{2.001^2}{2}} + \sqrt{\frac{\pi}{2}} * erf(\frac{2.001}{\sqrt{2}}) + 2 * e^{-\frac{2^2}{2}} - \sqrt{\frac{\pi}{2}} erf(\frac{2}{\sqrt{2}}) \right]$$

$$\frac{1}{\sqrt{2\pi}} [-0.2702647 + 1.19642 + 0.2706705 - 1.19629]$$

$$= \frac{1}{\sqrt{2\pi}} * 5.358 * 10^{-4}$$

Probability = $2.14*10^{-4}$

Dr. Gordon's Problems :

1. a. $E = 10^6$ V/cm ;

   Intensity = $\varepsilon_0 c E^2$ $(J^{-1}C^2m^{-1})$ (ms^{-1}) (V^2m^{-2})
   $= 8.854*10^{-12} * 3*10^8 * (10^8)^2$
   $= 2.66*10^{13}$ $Js^{-1}m^{-2}$
   $= 2.66*10^9$ W/cm^2

   b. Electric field felt by 1s electron = $q/4\pi\varepsilon_0 a_0^2$

   $$= \frac{1.6*10^{-19}}{4\pi*8.854*10^{-12}*(0.53*10^{-10})^2} = 5.12*10^{11} V/m$$

   Intensity = $\varepsilon_0 c E^2$
   $= 8.854*10^{-12} * 3*10^8 * (5.12*10^{11})^2$
   $= 6.96*10^{20}$ W/m^2
   $= 6.96*10^{16}$ W/cm^2

2. a. 3 + 2i :

   $$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$$

   $$\tan\theta = \frac{y}{x} = \frac{2}{3}$$

   $$\Rightarrow \theta = 33.69^o$$

   Therefore,
   3 + 2i = $\sqrt{13}$ e$^{i(33.69)}$

   b. 1 + i/2 – i :

   $$\frac{1+i}{2-i} * \frac{2+i}{2+i} = \frac{1+3i}{5} = \frac{1}{5} + \frac{3}{5}i$$

   $$= \Rightarrow r = \sqrt{(\frac{1}{5})^2 + (\frac{3}{5})^2} = \sqrt{\frac{10}{25}} = \sqrt{\frac{2}{5}}$$

   $$\theta = \tan^{-1}(\frac{3/5}{1/5}) = 71.57^o$$

   Therefore,

   1 + i/2 – i = $\sqrt{\frac{2}{5}}$ e$^{i(71.57)}$

3. f(v) = $Cv^2 \exp(-v^2/\alpha^2)$

$$\int_0^\infty Cv^2 e^{-\frac{v^2}{\alpha^2}} dv = 1$$

$$\Rightarrow C \cdot \frac{1}{2^2\sqrt{\frac{1}{\alpha^2}}}\sqrt{\pi\alpha^2} = 1$$

$$\Rightarrow C = \frac{4}{\sqrt{\pi}\alpha^3}$$

a. $$\langle v \rangle = \int_0^\infty v \cdot f(v) dv = \int_0^\infty v \cdot \frac{4}{\sqrt{\pi}\alpha^3} v^2 e^{-\frac{v^2}{\alpha^2}} dv = \frac{4}{\sqrt{\pi}\alpha^3}\int_0^\infty v^3 e^{-\frac{v^2}{\alpha^2}} dv = \frac{4}{\sqrt{\pi}\alpha^3}\frac{1}{2(\frac{1}{\alpha^2})^2}$$

i.e.

$$\langle v \rangle = \frac{2}{\sqrt{\pi}}\alpha = \mathbf{1.13\ \alpha}$$

b. v_{mp} :

$$\frac{d}{dv}(f(v)) = 0 \text{ at } v = v_{mp}$$

i.e. $$\frac{4}{\sqrt{\pi}\alpha^3}\frac{d}{dv}\left(v^2 e^{-\frac{v^2}{\alpha^2}}\right) = 0$$

$$= \frac{4}{\sqrt{\pi}\alpha^3}\left(2ve^{-\frac{v^2}{\alpha^2}} + v^2 e^{-\frac{v^2}{\alpha^2}} \cdot -\frac{2v}{\alpha^2}\right) = 0$$

$$= \frac{8}{\sqrt{\pi}\alpha^3} ve^{-\frac{v^2}{\alpha^2}}\left(1 - \frac{v^2}{\alpha^2}\right) = 0$$

$$\Rightarrow 1 - \frac{v^2}{\alpha^2} = 0 \Rightarrow v = \alpha$$

$\mathbf{v_{mp}} = \alpha$

c. v_{med} :

$$\int_0^{v_{med}} \frac{4}{\sqrt{\pi}\alpha^3} v^2 e^{-\frac{v^2}{\alpha^2}} dv = 0.5$$

$$\left(-\frac{2}{\sqrt{\pi}} x e^{-x^2} \Big|_0^{\frac{v_{med}}{\alpha}} + erf(\frac{v_{med}}{\alpha}) \right) = 0.5$$

$$\Rightarrow -\frac{2}{\sqrt{\pi}} \frac{v_{med}}{\alpha} e^{-\frac{v_{med}}{\alpha}^2} + erf(\frac{v_{med}}{\alpha}) = 0.5$$

Using trial & error, we find that $v_{med} = 1.09\ \alpha$ to make the above integral equal to 0.5. Therefore,

$v_{med} = 1.09\ \alpha$

d. $\sigma^2 = \langle v^2 \rangle - \langle v \rangle^2$

$$\langle v^2 \rangle = \frac{4}{\sqrt{\pi}\alpha^3} \int_0^\infty v^4 e^{-\frac{v^2}{\alpha^2}} dv = \frac{4}{\sqrt{\pi}\alpha^3} \frac{1 \cdot 3}{2^3 (\frac{1}{\alpha^2})^2} \sqrt{\pi\alpha^2} = \frac{3}{2}\alpha^2$$

Therefore,

$$\sigma^2 = \langle v^2 \rangle - \langle v \rangle^2 = 1.5\alpha^2 - (1.13\alpha)^2$$

σ = 0.47 α

4. a. P(v>v_{mean}) :

$$\int_{v_{mean}}^\infty \frac{4}{\sqrt{\pi}\alpha^3} v^2 e^{-\frac{v^2}{\alpha^2}} dv = \int_0^\infty \frac{4}{\sqrt{\pi}\alpha^3} v^2 e^{-\frac{v^2}{\alpha^2}} dv - \int_0^{v_{mean}} \frac{4}{\sqrt{\pi}\alpha^3} v^2 e^{-\frac{v^2}{\alpha^2}} dv$$

$$= 1 - \left(-\frac{2}{\sqrt{\pi}} x e^{-x^2} \Big|_0^{\frac{2}{\sqrt{\pi}}} + erf(\frac{2}{\sqrt{\pi}}) \right)$$

$$= 1 + \frac{4}{\pi} e^{-\frac{4}{\pi}} - erf(1.13)$$

$$= 1 + 0.3564 - 0.88997 = 0.4664$$

P(v>v_{mean}= $\frac{2}{\sqrt{\pi}}\alpha$) = 0.4664

b. P(v>v_{mp}) :

$$\text{Probability} = \begin{array}{l} 1 - \left(-\frac{2}{\sqrt{\pi}} x e^{-x^2} \Big|_0^1 + erf(1) \right) \\ = 1 + \frac{2}{\sqrt{\pi}} e^{-1} - erf(1) \end{array}$$

$= 1 + 0.4151 - 0.84270$
$= 0.5724$

P(v>v$_{mp}$= α) = 0.5724

c. P(v>v$_{med}$) :

$$\text{Probability} = 1 - \left(-\frac{2}{\sqrt{\pi}} x e^{-x^2} \Big|_0^{1.09} + erf(1.09) \right)$$

$$= 1 + \frac{2}{\sqrt{\pi}} * 1.09 * e^{-1.19} - erf(1.09)$$

$= 1 + 0.3749 - 0.87680$
$= 0.4981$

P(v>v$_{med}$= 1.09α) = 0.4981

d. P(v>v$_{mean}$ + σ) :

$v_{mean} + \sigma = 1.13\ \alpha + 0.47\ \alpha = 1.6\ \alpha$

$$\text{Probability} = 1 - \left(-\frac{2}{\sqrt{\pi}} x e^{-x^2} \Big|_0^{1.6} + erf(1.6) \right)$$

$$= 1 + \frac{2}{\sqrt{\pi}} * 1.6 * e^{-2.56} - erf(1.6)$$

$= 1 + 0.1396 - 0.97635$
$= 0.1632$

P(v>v$_{mean}$ + σ = 1.6α) = 0.1632

e. P(v>v$_{mean}$ - σ) :

$v_{mean} - \sigma = 1.13\ \alpha - 0.47\ \alpha = 0.66\ \alpha$

$$\text{Probability} = 1 - \left(-\frac{2}{\sqrt{\pi}} x e^{-x^2} \Big|_0^{0.66} + erf(0.66) \right)$$

$$= 1 + \frac{2}{\sqrt{\pi}} * 0.66 * e^{-0.44} - erf(0.66)$$

$= 1 + 0.4818 - 0.64938 = 0.8324$

P(v>v$_{mean}$ - σ = 0.66α) = 0.8324