Lecture 18. Introduction to Angular Momentum

Classical examples:

1. Orbital angular momentum
   L = pb
   b = impact parameter
   p = linear momentum
   L = $\mu v b$

2. Rotation in the *xy*-plane plane about a point: $\hat{L} = rp\hat{k}$

3. General definition:

$$\vec{L} = \vec{r} x \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \left(yp_z - zp_y\right)\hat{i} + \left(zp_x - xp_z\right)\hat{j} + \left(xp_y - yp_x\right)\hat{k} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$$

$$L^2 = L_x^2 + L_y^2 + L_z^2$$

Cyclic permutation of $x \rightarrow y \rightarrow z$ results in $L_x \rightarrow L_y \rightarrow L_z$

Quantum mechanical operators:

$$\hat{L}_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Commutators:

$$
\begin{aligned}
\left[\hat{L}_x, \hat{L}_y\right] &= \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x \\
&= \left(y\hat{p}_z - z\hat{p}_y\right)\left(z\hat{p}_x - x\hat{p}_z\right) - \left(z\hat{p}_x - x\hat{p}_z\right)\left(y\hat{p}_z - z\hat{p}_y\right) \\
&= y\hat{p}_z z\hat{p}_x - \left\{y\hat{p}_z x\hat{p}_z + z\hat{p}_y z\hat{p}_x\right\} + z\hat{p}_y x\hat{p}_z \\
&\quad - z\hat{p}_x y\hat{p}_z + \left\{z\hat{p}_x z\hat{p}_y + x\hat{p}_z y\hat{p}_z\right\} - x\hat{p}_z z\hat{p}_y \\
&= y\hat{p}_z z\hat{p}_x + z\hat{p}_y x\hat{p}_z - z\hat{p}_x x\hat{p}_z - x\hat{p}_z z\hat{p}_y \\
&= \hat{p}_z z y\hat{p}_x - z\hat{p}_z y\hat{p}_x + z\hat{p}_z x\hat{p}_y - \hat{p}_z z x\hat{p}_y \\
&= [\hat{p}_z, z] y\hat{p}_x + [z, \hat{p}_z] x\hat{p}_y \\
&= -[z, \hat{p}_z] y\hat{p}_x + [z, \hat{p}_z] x\hat{p}_y \\
&= i\hbar\left(x\hat{p}_y - y\hat{p}_x\right) \\
&= i\hbar\hat{L}_z
\end{aligned}
$$

Cyclic permutation:

$$
\left[\hat{L}_x, L_y\right] = i\hbar\hat{L}_z \rightarrow \left[\hat{L}_y, L_z\right] = i\hbar\hat{L}_x \rightarrow \left[\hat{L}_z, L_x\right] = i\hbar\hat{L}_y
$$

What about the squared operators?

$$
\left[\hat{L}^2, \hat{L}_x\right] = \left[\hat{L}_x^2, \hat{L}_x\right] + \left[\hat{L}_y^2, \hat{L}_x\right] + \left[\hat{L}_z^2, \hat{L}_x\right]
$$

$$
\left[\hat{L}_x^2, \hat{L}_x\right] = 0
$$

$$
\begin{aligned}
\left[\hat{L}_y^2, \hat{L}_x\right] &= \hat{L}_y^2\hat{L}_x - \hat{L}_x\hat{L}_y^2 \\
&= \hat{L}_y^2\hat{L}_x - \hat{L}_y\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x\hat{L}_y - \hat{L}_x\hat{L}_y^2 \\
&= \hat{L}_y\left[\hat{L}_y, \hat{L}_x\right] + \left[\hat{L}_y, \hat{L}_x\right]\hat{L}_y \\
&= -i\hbar\hat{L}_y\hat{L}_z - i\hbar\hat{L}_z\hat{L}_y
\end{aligned}
$$

Similarly, you can show that

$$
\left[\hat{L}_z^2, \hat{L}_x\right] = i\hbar\hat{L}_z\hat{L}_y + i\hbar\hat{L}_y\hat{L}_z
$$

It follows that $\left[\hat{L}^2, L_x\right] = 0$

Here is a physical explanation of the incompatibility of all the components of the angular momentum vector. Imagine a particle moving in a circular orbit with angular momentum L. Suppose you knew all three components of the L precisely. For example, suppose L_z has some definite value, and that $L_x = L_y = 0$. In other words, suppose L points along the z-axis precisely. Then the particle is confined to the xy plane. Then you know that z and p_z are both zero. This violates the uncertainty principle.

Cyclically: $\left[\hat{L}^2, L_y\right] = 0 = \left[\hat{L}^2, L_z\right]$

We choose L^2 and L_z as a compatible pair of observables..

Having determined what are the compatible operators, we now ask what are their eigenfunctions and eigenvalues. That is, we wish to solve the following equations:

$$\hat{L}_z Y = bY$$
$$\hat{L}^2 Y = cY$$

The natural coordinates to use are spherical polar coordinates.

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$Y = Y(\theta,\phi)$$

In these coordinates, the operators become

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$
$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

Since $\hat{L}_z$ is depends only on ϕ, and since Y is an eigenfunction of both $\hat{L}_z$ and $\hat{L}^2$, it follows that Y must be separable in θ and ϕ:

$$Y(\theta,\phi) = S(\theta)T(\phi)$$

Normalization:

$$\int_0^{\pi} \left|S(\theta)\right|^2 \sin\theta d\theta = 1$$
$$\int_0^{2\pi} \left|T(\phi)\right|^2 d\phi = 1$$

Boundary conditions:

$$S(\theta) = S(\theta + \pi)$$
$$T(\phi) = T(\phi + 2\pi)$$

Eigenfunctions and eigenvalues of $\hat{L}_z$

$$-i\hbar \frac{dT}{d\phi} = bT(\phi)$$
$$\frac{dT}{T} = ib\hbar d\phi$$
$$T(\phi) = Ae^{ib\phi/\hbar}$$

Boundary condition:

$$Ae^{ib\phi/\hbar} = Ae^{ib(\phi+2\pi)/\hbar}$$
$$1 = e^{2i\pi b/\hbar}$$

But $e^{2i\pi} = 1 \Rightarrow b/\hbar = m$, where m=0 or an integer.

$$T(\phi) = Ae^{im\phi}$$
$$\hat{L}T(\phi) = m\hbar T(\phi)$$

Normalization:

$$\int_0^{2\pi} |T_m(\phi)|^2 = 1$$
$$2\pi |A|^2 = 1$$
$$|A| = \frac{1}{\sqrt{2\pi}}$$
$$T_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

All ϕ values are equally probable.

Orthogonality:

$$\int_0^{2\pi} T_m(\phi) T_n(\phi) d\phi = \delta_{mn}$$

Eigenfunctions and eigenvalues of $\hat{L}^2$

$$\hat{L}^2 ST = cST$$
$$\hat{L}^2 e^{im\phi} S(\theta) = ce^{im\phi} S(\theta)$$

$$= -\hbar^2 \left(\frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right) S(\theta) e^{im\phi}$$
$$= -\hbar^2 \left(S'' + \cot\theta S' - \frac{m^2}{\sin^2\theta} S \right)$$

Canceling out the ϕ dependence (which is separable!), we get the equation

$$S'' + \cot\theta S' - \frac{m^2}{\sin^2\theta} S = -\frac{c}{\hbar^2} S$$

Transformation of variable:

$$x \equiv w \equiv \cos\theta,$$
$$-1 \le w \le 1$$
$$G(w) = S(\theta)$$
$$dw = -\sin\theta d\theta$$

After expressing the first and second derivatives of S in terms of the derivatives of G, we readily get

$$\left(1 - w^2\right) G'' - 2wG' + \left[\frac{c}{\hbar^2} - \frac{m^2}{1 - w^2} \right] G = 0.$$

This equation is identical to the associated Legendre equation if we make the identification

$$c = l(l+1)\hbar^2,$$

where *l* is zero or a positive integer.

Show that is a Sturm-Liouville equation. Its solutions are the associated Legendre polynomials,

$$S_{l,m}(\theta) = \left[\left(\frac{2l+1}{2}\right)\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2} P_l^{|m|}(\cos\theta)$$

Orthogonality:

$$\int_0^{\pi} S_l^m(\theta) S_k^m(\theta) \sin\theta d\theta = \int_{-1}^{1} G_l^m(x) G_k^m(x) dx = \delta_{kl}$$

Note what this condition does *not* say.

Generating functions:

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l$$

$$P_l^m(x) = \frac{\left(1-x^2\right)^{m/2}}{2^l l!}\frac{d^{l+m}}{dx^{l+m}}\left(x^2-1\right)^l$$

Recursion relations:

$$(2l+1)P_l(x) = \frac{d}{dx}\left(P_{l+1} - P_{l-1}\right)$$

$$(l+1)P_l(x) = \frac{dP_{l+1}}{dx} - x\frac{dP_l}{dx}$$

$$xP_l^{|m|}(x) = \cos\theta P_l^{|m|}(\cos\theta) = \frac{1}{2l+1}\left\{(l-|m|+1)P_{l+1}^{|m|} + \left(l+|m|\right)P_{l-1}^{|m|}\right\}$$

$$\sqrt{1-x^2}\,P_l^{|m|}(x) = \sin\theta P_l^{|m|}(\cos\theta)$$

$$= \frac{1}{2l+1}\left\{P_{l+1}^{|m|+1} + P_{l-1}^{|m|}\right\} = \frac{1}{2l+1}\left\{\left(l+|m|\right)\left(l+|m|-1\right)P_{l-1}^{|m|-1} - \left(l-|m|\right)\left(l-|m|+2\right)P_{l+1}^{|m|-1}\right\}$$

Lecture 19. Model Problems Involving Angular Momentum

Particle on a Ring

$$\hat{L} = \hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$$

$$\hat{H} = \frac{\hat{L}_z^2}{2\mu R^2} = \frac{\hat{L}_z^2}{2I} = -\frac{\hbar^2}{2I}\frac{d^2}{d\phi^2}$$

$$T_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

$$\hat{H}T_m = \frac{m^2\hbar^2}{2I} T_m$$

$$E_m = \frac{m^2\hbar^2}{2I}$$

There is no zero-point energy. Levels are spaced quadradically and are two-fold degenerate (except for m = 0).

Free Rotor

In this case we must use the full three-dimensional angular momentum.

$$\hat{H} = \frac{\hat{L}^2}{2I}$$

$$Y_{lm}(\theta,\phi) = S_{l,|m|}(\theta,\phi) T_m(\phi) \propto Y_{l,|m|}(\theta,\phi) e^{im\phi}$$

$$\hat{H}Y_{lm} = \frac{l(l+1)\hbar^2}{2I} Y_{lm}$$

In dealing with rotation, it is conventional to label the quantum number by *J* rather than *l*.

The energy levels are again spaced quadratically, and here the degeneracy is 2J+1.

The length of the J vector is given by $\sqrt{J(J+1)}\hbar$ For large J,

$$\sqrt{J(J+1)} = J\sqrt{1+\frac{1}{J}} \approx J\left(+\frac{1}{2}\frac{1}{J}\right) = J + \frac{1}{2}$$

The angle between the angular momentum vector and the z-axis is given by

$$\cos\alpha = \frac{M_J}{\sqrt{J(J+1)}}$$

The same formulism describes a particle confined to move on the surface of a sphere.

What are the expectation values of the components of $\vec{J}$?

We already know that $\langle J_z \rangle = M_J \hbar$. This means that $\langle J_z^2 \rangle = M_J^2 \hbar^2$. It follows that

$$\langle J_x^2 + J_y^2 \rangle = [J(J+1) - M_J^2]\hbar^2$$

By symmetry $\langle J_x^2 \rangle = \langle J_y^2 \rangle$

Later we will prove than $\langle J_x \rangle = \langle J_y \rangle = 0.$

It follows that the polar angle of $\vec{J}$ is completely unknown. This leads to a physical picture of the $\vec{J}$ vector precessing about the z-axis at an angle of inclination equal to α.

Physical picture of the rotor:

1. Suppose M_J = J. This means that J is roughly parallel to the z-axis, and the molecule rotates in the *xy*-plane. The azimuthal angle of $\vec{J}$ is given by

$$\cos\alpha = \frac{M_J}{\sqrt{2}} = \frac{J}{\sqrt{2}}$$

For J =1, $Y_{11} \propto \sin\theta$, $\alpha = 45^0$, and the probability density is concentrated near the equator. The same picture applies for M_J = -J, except that the molecule rotates in the opposite direction. The probability distribution is given by

$$p(\theta)d\theta = |Y_{JM}(\theta,\phi)|^2 \sin\theta d\theta \propto \sin^3\theta$$

The most probable θ is given by

$$\frac{dp}{d\theta} \propto \sin^2\theta\cos\theta = 0 \Rightarrow \theta = \pi/2$$

2. Suppose M_J = 0. In this case, J lies in the xy-plane, $\alpha = 0,$ and the molecule rotates in a plane containing the z axis. This plane precesses about the z-axis because the direction of J in the xy plane is completely unknown. For J =1, $Y_{10} \propto \cos\theta$, and the probability density is concentrated near the poles.

$$p(\theta)d\theta = |Y_{JM}(\theta,\phi)|^2 \sin\theta d\theta \propto \cos^2\theta\sin\theta$$

The most probable θ is given by

$$\frac{dp}{d\theta} \propto \cos^3\theta - 2\sin^2\theta\cos\theta = 0$$

$$\tan^2\theta = 1/2 \Rightarrow \theta = 35.3^0$$

Note that the angle of the greatest probability density is not necessarily the most probable angle. This is where the Jacobian is crucial.

Practical considerations for a rotating linear molecule:

The energy levels are given by $E_J = \frac{J(J+1)\hbar^2}{2\mu R^2} = hcBJ(J+1),$ where B is called the rotational constant and has the units of wave numbers.

Statistical mixtures: The Boltzmann population is given as usual by

$$P_J = \frac{g_J e^{-E_J/kT}}{\sum_J g_{J'} e^{-E_J/kT}}$$

where the degeneracy is $g_J = 2J+1.$

The denominator is called a partition function and has the value

$$Q = \sum_J (2J+1)e^{-hcBJ(J+1)/kT} \approx \int_0^\infty e^{-hcBJ(J+1)/kT}(2J+1)dJ = \frac{kT}{hcB}$$

The population of level J is therefore

$$P_J \approx \frac{hcB}{kT}(2J+1)e^{-hcBJ(J+1)/kT}$$

The most probable J is calculated by taking the maximum of P_J:

$$\frac{dP_J}{dJ} = 0$$
$$2 - \frac{hcB}{kT}(2J_m+1)^2 = 0$$
$$J_m = \sqrt{\frac{kT}{2hcB}} - \frac{1}{2}$$

Remember that kT at 300 K is 208 cm^{-1}. Here are some examples at 300 K:

Molecule	B(cm^{-1})	P_0	J_m	P_{J_m}
H_2	60.8	0.29	1	0.49
I_2	0.037	0.00018	53	0.011

We can also describe a rotational wave packet and pendular states.

$$\psi(\theta,\phi,t)=\sum_J c_J Y_{J,M_J}(\theta,\phi)e^{-E_J t/\hbar}$$

Prove that the (full) revival time of such a wave packet is given by $\frac{1}{2Bc}$

Alignment.

We define the alignment of a molecule by

$$A=\langle\cos^2\theta\rangle=\int_0^{\pi}\int_0^{2\pi}Y_{JM}^{*}(\theta,\phi)\cos^2\theta Y_{JM}(\theta,\phi)\sin\theta d\theta d\phi$$

If you don't remember the normalization constant, you can work it out at the same time:

$$A=\frac{\int_0^{\pi}\left|P_J^M(\cos\theta)\right|^2\cos^2\theta\sin\theta d\theta}{\int_0^{\pi}\left|P_J^M(\cos\theta)\right|^2\sin\theta d\theta}=\frac{\int_{-1}^{1}\left|P_J^M(x)\right|x^2dx}{\int_{-1}^{1}\left|P_J^M(x)\right|^2dx}$$

J=1, M=0

$$A=\frac{\int_{-1}^{1}x^4dx}{\int_{-1}^{1}x^2dx}=\frac{2/5}{2/3}=3/5$$

J=1, M=1,-1

$$A=\frac{\int_{-1}^{1}(1-x^2)x^2dx}{\int_{-1}^{1}(1-x^2)dx}=\frac{2/3-2/5}{2-2/3}=1/5$$

$$A=\left(\frac{1}{3}\right)\left(\frac{3}{5}\right)+\left(\frac{2}{3}\right)\left(\frac{1}{5}\right)=\frac{1}{3}$$

Lecture 20. Ladder Operators

Definition of the ladder operators:

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y$$
$$\hat{L}_- = \hat{L}_x - i\hat{L}_y$$

Commutation properties:

$$\hat{L}_+\hat{L}_- = \left(\hat{L}_x + i\hat{L}_y\right)\left(\hat{L}_x - i\hat{L}_y\right) = \hat{L}_x^2 + \hat{L}_y^2 + i\left(\hat{L}_y\hat{L}_x - \hat{L}_x\hat{L}_y\right) = \hat{L}_x^2 + \hat{L}_y^2 - i\left[\hat{L}_x, \hat{L}_y\right] = \hat{L}_x^2 + \hat{L}_y^2 + \hbar\hat{L}_z = \hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z$$

Prove the following:

$$\left[\hat{L}_+, \hat{L}_-\right] = 2\hbar\hat{L}_z$$
$$\left[\hat{L}_z, \hat{L}_\pm\right] = \hbar\hat{L}_\pm$$
$$\left[\hat{L}^2, \hat{L}_\pm\right] = 0$$

Suppose that the following are true:

$$\hat{L}_z Y = bY$$
$$\hat{L}^2 Y = cY$$

Question: Is $\hat{L}_+ Y$ also an eigenfucntion of $\hat{L}_z$ and $\hat{L}^2$? Find out by operating on the function:

$$\hat{L}_z\left(\hat{L}_+ Y\right) = (b+\hbar)\left(\hat{L}_+ Y\right)$$
$$\hat{L}^2\left(\hat{L}_+ Y\right) = c\left(\hat{L}_+ Y\right)$$

The first result follows from the commutator property, $\hat{L}_z\hat{L}_+ = \hat{L}_+\hat{L}_z + \hbar\hat{L}_+$.
Conclusion: It is indeed an eigenfunction of both, with the same eigenvalue of $\hat{L}^2$ but with an eigenvalue of $\hat{L}_z$ that is increased by $\hbar$. That is why $\hat{L}_+$ is called a raising operator.

By induction we deduce that

$$\hat{L}_z\left(\hat{L}_+^n Y\right) = (b+n\hbar)\left(\hat{L}_+^n Y\right)$$
$$\hat{L}^2\left(\hat{L}_+^n Y\right) = c\left(\hat{L}_+^n Y\right)$$

We can also prove that $\hat{L}_-$ is a lowering operator. Prove the following:

$$\hat{L}_z\left(\hat{L}_- Y\right) = (b-\hbar)\left(\hat{L}_+ Y\right)$$
$$\hat{L}^2\left(\hat{L}_- Y\right) = c\left(\hat{L}_- Y\right)$$

And by induction:

$$\hat{L}_z\left(\hat{L}_-^n Y\right) = (b-n\hbar)\left(\hat{L}_-^n Y\right)$$
$$\hat{L}^2\left(\hat{L}_-^n Y\right) = c\left(\hat{L}_-^n Y\right)$$

Question: Is there any bound on n?

$$\hat{L}_z^2\left(\hat{L}_\pm Y\right) = \hat{L}_z\left[\hat{L}_z\left(\hat{L}_\pm Y\right)\right] = \hat{L}_z\left[(b\pm\hbar)\hat{L}_\pm Y\right] = (b\pm\hbar)^2\,\hat{L}_\pm Y$$

By induction:

$$\hat{L}_z^2\left(\hat{L}_\pm^n Y\right) = (b\pm n\hbar)^2\,\hat{L}_\pm^n Y$$

But $\left\langle \hat{L}_z^2 \right\rangle \le \left\langle \hat{L}^2 \right\rangle$. Therefore, $(b\pm n\hbar)^2 \le c$.
We conclude that

$$b_{\max} \le \sqrt{c} \ \text{ with eigenfunction } Y_{\max}$$

$$b_{\min} \ge -\sqrt{c} \ \text{ with eigenfunction } Y_{\min}$$

That is,

$$\hat{L}_+ Y_{\max} = 0$$
$$\hat{L}_- Y_{\min} = 0$$

The reason is that

$$\hat{L}_z\left(\hat{L}_+ Y_{\max}\right) = \left(b_{\max} + \hbar\right)\left(\hat{L}_+ Y_{\max}\right)$$

But this cannot be true unless one of the factors is zero.

It therefore follows that $\hat{L}_-\hat{L}_+ Y_{\max} = 0$.

But we already know that $\hat{L}_-\hat{L}_+ Y = \left(\hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z\right)Y$

It follows that

$$\left(c - b_{\max}^2 - \hbar b_{\max}\right) Y_{\max} = 0$$
$$c - b_{\max}^2 - \hbar b_{\max} = 0$$
$$c = b_{\max}^2 + \hbar b_{\max}$$

Similarly,

$$\hat{L}_+ \hat{L}_- Y_{\min} = 0 \Rightarrow c = b_{\min}^2 - \hbar b_{\min}$$

Therefore

$$b_{\max}^2 + \hbar b_{\max} = b_{\min}^2 - \hbar b_{\min}$$

Solving the quadratic equation for b_{max} gives

$$b_{\max} = \frac{1}{2}\left[-\hbar \pm \left(\hbar - 2b_{\min}\right)\right]$$
$$= -b_{\min} \text{ or } b_{\min} - \hbar$$

We reject the second solution because we know that $b_{max} > b_{min}$. Therefore $b_{\max} = -b_{\min}$

Now suppose that we apply the lowering operator N times:

$$Y_{\min} \propto \hat{L}_-^N Y_{\max} \Rightarrow b_{\max} - b_{\min} = N\hbar$$

Combining these two results gives $b_{\min} = -N\hbar/2$ and $b_{\max} = N\hbar/2$

Setting $l = N/2$ gives $b_{\max} = l\hbar$ and $b_{\min} = -l\hbar$, where J is an integer or half integer.

We also conclude that $c = b_{\max}^2 + \hbar b_{\max} = l(l+1)\hbar^2$.

What are the expectation values of $\hat{L}_x$ and $\hat{L}_y$? From the definitions of $\hat{L}_+$ and $\hat{L}_-$ we readily show that

$$\hat{L}_x = \frac{1}{2}\left(\hat{L}_+ + \hat{L}_-\right)$$
$$\hat{L}_y = \frac{1}{2i}\left(\hat{L}_+ - \hat{L}_-\right)$$

But $\langle lm|\hat{L}_\pm|lm\rangle \propto \langle lm|l, m \pm 1\rangle = 0 \Rightarrow \langle \hat{L}_x \rangle = \langle \hat{L}_y \rangle = 0.$

What are the expectation values of $\hat{L}_x^2$ and $\hat{L}_y^2$?

$$\hat{L}_x^2 = \frac{1}{4}\left(\hat{L}_+ + \hat{L}_-\right)\left(\hat{L}_+ + \hat{L}_-\right) = \frac{1}{4}\left(\hat{L}_+^2 + \hat{L}_-^2 + \hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+\right) = \frac{1}{4}\hat{L}_+^2 + \frac{1}{4}\hat{L}_-^2 + \frac{1}{4}\left(\hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z\right) + \frac{1}{4}\left(\hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z\right)$$
$$= \frac{1}{4}\hat{L}_+^2 + \frac{1}{4}\hat{L}_-^2 + \frac{1}{2}\left(\hat{L}^2 - \hat{L}_z^2\right)$$

It follows that

$$\left\langle \hat{L}_x^2 \right\rangle = \frac{1}{2}\left\langle \hat{L}^2 \right\rangle - \frac{1}{2}\left\langle \hat{L}_z^2 \right\rangle = \frac{1}{2}\hbar^2\left[l(l+1) - m^2\right] = \left\langle \hat{L}_y^2 \right\rangle$$

Now let's solve for the eigenfunctions. We start by invoking separability in θ and ϕ:

$$Y_{ll}(\theta,\phi) = \frac{1}{\sqrt{2\pi}} e^{il\phi} S_{ll}(\theta)$$

The equation we wish to solve is

$$\hat{L}_+ Y_{ll} = 0.$$

Using the transformation from Cartesian to polar coordinates, you can show that

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y = \hbar e^{\pm il\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$

Our equation is therefore

$$e^{il\phi} S'_{ll} + i\cot\theta S_{ll}(ile^{il\phi}) = 0$$
$$S'_{ll} = l\cot\theta S_{ll}$$

You can verify that the solution is

$$S_{ll}(\theta) = C_{ll}\sin^l\theta$$

Proof: $\frac{d}{d\theta}\left(\sin^l\theta\right) = l\left(\sin^{l-1}\theta\right)\cos\theta = l\cot\theta\sin^l\theta$

You can work out the normalization and get

$$Y_{ll}(\theta,\phi) = \frac{1}{\sqrt{2\pi}} e^{il\phi}(-1)^L \sqrt{\frac{(2l+1)!}{2}}\frac{1}{2^l l!}\sin^l\theta$$

Finally, we can apply the lowering operator to get all the other eigenfunctions:

$$Y_{lm} = C_{lm}\hat{L}_-^{l-m}Y_{ll} = C_{lm}\left[-\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)\right]^{l-m} Y_{ll}$$

What are the coefficients? Let

$$\hat{L}_+|lm\rangle = C_+|l,m+1\rangle$$

When we learn about Hermitian operators, we will be able to show that

$$\langle lm|\hat{L}_- = \langle l,m+1|C_+^*$$

It follows that

$$\langle lm|\hat{L}_-\hat{L}_+|lm\rangle = \langle l,m+1|C_+^*C_+|l,m+1\rangle = |C_+|^2$$

But

$$\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hat{L}_z\hbar$$

Therefore

$$\langle lm|L^2 - \hat{L}_z^2 - \hat{L}_z\hbar|lm\rangle = [l(l+1)+m^2-m]\hbar^2$$

By sign (and phase convention),

$$C_+ = [l(l+1)-m(m+1)]^{1/2}\hbar$$

Similarly,

$$C_- = [l(l+1)-m(m-1)]^{1/2}\hbar$$